# Time dependent quantum generators for the Galilei group

Gianluigi Filippelli

Prague, 16th June, 2007

# Motivation

Doebner and Mann introduced an approach to the ray representations of the Galilei group in $(1 + 1)$-dimensions, giving rise to quantum genertors with an explicit dependence on time.
Our purpouse is to extend their approach to higher dimensions. As a result, we determine the generators of the ray representation in $(2 + 1)$- and $(3 + 1)$-dimensions.

1 Introduction
- Wigner's Theorem and Ray Representations
- Imprimitivity system
- Phase exponents

2 Galilei's group
- Galilei's group in $(3+1)$-dimensions
- Galilei's group in $(2+1)$-dimensions
- Galilei's group in $(1+1)$-dimensions
- Extension of Doebner and Mann approach

# Wigner's Theorem and Ray Representations

$\mathcal{R}_\psi = \{\varphi \in \mathcal{H} : |\varphi\rangle = \tau\, |\psi\rangle\, , |\tau| = 1\} \in \mathcal{R}$, where $\mathcal{R}$ set of rays in Hilbert space.

## Wigner's Theorem[1]

*$\mathcal{H}$ Hilbert space. $T$ symmetry transformation, $T : \mathcal{R} \to \mathcal{R}$, $T\mathcal{R}_\psi = \mathcal{R}_{\psi'}$, such that*

$$|\,\langle \psi_1 \,|\, \psi_2 \rangle\,| = |\,\langle \psi_1' \,|\, \psi_2' \rangle\,|$$

*$\exists U$, unitary or antiunitary, such that $\psi' = U\psi$, univocally determined up a phase factor.*

$$G \ni r \to \mathcal{U}_r = \{U_r', r \in G : U_r' = \phi(r) U_r, |\phi(r)| = 1\}$$

$\mathcal{U}_r \hookrightarrow U_r,\, U_r U_s \neq U_{rs},\, U_r U_s \in \mathcal{U}_{rs} \Rightarrow U_r U_s = \omega(r,s) U_{rs}$

## Ray Representation[2]

$G$ Lie group. $r, s, t, \ldots \in G$.

$$R(G) : r \to U_r \in \mathcal{U}_r$$
$$\mathcal{U}_r \mathcal{U}_s = \mathcal{U}_{rs} \qquad \to \qquad U_r U_s = \omega(r,s) U_{rs}$$

$|\omega(r,s)| = 1$, con $\omega(r,s)$ phase factor or multiplicator.

[1] Wigner, 1959; Bargmann, 1964; Weinberg, 1995

[2] Bargmann, 1954

# Localizable physical system in $\mathbb{R}$

$G$ translation group on $\mathbb{R}$. $\Delta \to E(\Delta)$

### Definition

$(\mathcal{H}, E, U)$, where $\mathcal{H}$ Hilbert space, $E$ projection valued measure $E(\Delta)$ on $\mathcal{H}$, $U : G \to \mathcal{U}$ representation of $G$, such that

$$E(\Delta) = U_a^{-1} E(\Delta - a) U_a$$

it is called Imprimitivity System[3] based on $(\mathbb{R}, \mathscr{B}(\mathbb{R}), G)$.

### Von Neumann's Theorem[4]

$\mathcal{H}$ Hilbert space. $P$, $Q$ adjoint operators such that $U(\alpha) = \mathrm{e}^{i\alpha P}$, $V(\beta) = \mathrm{e}^{i\beta Q}$ satisfy the following:

$$U(\alpha)V(\beta) = \mathrm{e}^{i\alpha\beta} V(\beta)U(\alpha)$$

then $P$, $Q$ are a Schrödinger's couple, $[Q, P] = i$.

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[3] Mackey, 1952

[4] von Neumann, 1931; Putnam, 1967; Jauch, 1968; Weyl, 1931

# Localizable physical system in $\mathbb{R}$

$G$: group of Galilei's transformations
$G \ni r = (\alpha, v)$, with product defined by
$rs = (\alpha_r, v_r) \cdot (\alpha_s, v_s) = (\alpha_r + \alpha_s, v_r + v_s)$
Let $W(\alpha, v)$ be a representation of $G$, $W \in \mathcal{U}(\mathcal{H})$. Then:

$$W(\alpha_r, v_r) W(\alpha_s, v_s) = \mathrm{e}^{i\frac{k}{2}(\alpha_r v_s - \alpha_s v_r)} W(\alpha_r + \alpha_s, v_r + v_s)$$

$$\begin{cases} U_\alpha = W(\alpha, 0) \\ G_v^{-1} = W(0, v) \end{cases} \rightarrow U_\alpha G_v^{-1} = \mathrm{e}^{i\alpha\beta} G_v^{-1} U_\alpha \rightarrow \begin{cases} U_\alpha = \mathrm{e}^{i\alpha P} \\ G_v^{-1} = \mathrm{e}^{i\beta Q} \end{cases}$$

The generator of time translation operator coincides with the system's hamiltonian operator:

$$H = H_0 + V(Q) \quad \rightarrow \quad H\,|\psi\rangle = i\frac{\mathrm{d}}{\mathrm{d}\,t}\,|\psi\rangle$$

$$H_0 = \frac{P^2}{2k} \quad \rightarrow \quad k \leftrightarrow \text{mass}$$

# Phase factors and phase exponents

$G$ Lie group, $r, s, t \in G$, $e$ identity of $G$. Let be $\mathscr{N}$ a neighbourhood of $e$.

$$R(G) : r \longrightarrow U_r \in \mathcal{U}_r$$
$$U_r U_s = \omega(r,s) U_{rs}, \text{ with } r, s \in \mathscr{N}$$

Equivalence relation:

$$\omega'(r,s) = \frac{\phi(r)\phi(s)}{\phi(rs)}\omega(r,s), \qquad U'_r = \phi(r) U_r, \text{ with } |\phi(r)| = 1,$$

$\omega(r,s) = \mathrm{e}^{i\xi(r,s)}$ where $\xi(r,s)$ local phase exponent, $\forall r, s \in \mathscr{N} \subset G$

$$\left.\begin{aligned} \xi(r,e) &= \xi(e,r) = 0 \\ \xi(r,s) + \xi(rs,t) &= \xi(r,st) + \xi(s,t) \end{aligned}\right. \qquad \forall r,s,t \in \mathscr{N}$$

For connected and simply connected Lie groups:

$$\xi(r,s) \text{ local} \longrightarrow \xi(r,s) \text{ global}$$

# Phase exponents and infinitesimal exponents

- Equivalence relation: $\xi \sim \xi_0$ if

$$\xi(r,s) = \xi_0(r,s) + \zeta(r) + \zeta(s) - \zeta(rs) = \xi_0(r,s) + \Delta_{r,s}[\zeta]$$

- $s \in r(\tau) \Rightarrow \xi(r,s) = \xi(s,r) = 0$
- Local group: $\xi(r,s) \longrightarrow H : \{\theta, r\}, \theta \in \mathbb{R}, r \in G$

$\mathfrak{g}$ Lie algebra of group $G$.
$\exists\, \Xi(a,b)$, with $a, b \in \mathfrak{g}$, bilinear, anti-symmetric, such that

$$\Xi(a,b) = \lim_{\tau \to 0} \tau^{-2} \left(\xi\left((\tau a)(\tau b), (\tau a)^{-1}(\tau b)^{-1}\right) + \right.$$
$$\left. + \xi(\tau a, \tau b) + \xi\left((\tau a)^{-1}, (\tau b)^{-1}\right)\right)$$

$$\mathrm{d}\,\Xi(a,a',a'') = \Xi([a,a'],a'') + \Xi([a',a''],a) + \Xi([a'',a],a') = 0$$

with $\Xi$ infinitesimal exponent of $\mathfrak{g}$.

# Infinitesimal exponents

$\{a_n\}$ basis of $\mathfrak{g}$, $a = \sum \gamma^i a_i$, $a' = \sum {\gamma'}^i a_i$ $(1 \leqslant i \leqslant n)$

$$\Xi(a, a') = \sum_{ij} \beta_{ij} \gamma^i {\gamma'}^j, \quad \beta_{ij} = -\beta_{ji} = \Xi(a_i, a_j)$$

$$\beta_{ij} = \left( \frac{\partial^2 \xi(r,s)}{\partial \rho^i \partial \sigma^j} - \frac{\partial^2 \xi(r,s)}{\partial \rho^j \sigma^i} \right)_{\rho^k = \sigma^k = 0}$$

Equivalence relation:

$$\Xi'(a, b) = \Xi(a, b) - \Lambda([a, b]) = \Xi(a, b) - \mathrm{d}[\Lambda]$$

$\Lambda(a_i) = \lambda_i$, $\beta_{ij} = \sum {c^m}_{ij} \lambda_m$ $(1 \leqslant m \leqslant n)$

# Bargmann's Theorem

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1. On a Lie group $G$, every local exponent $\xi(r,s)$ is equivalent to a canonical local exponent $\xi'(r,s)$ which, on some canonical neighborhood $\mathscr{N}$, is analytic in the canonical coordinates $r, s$ and vanishes if $r, s$ belong to the same local one-parameter subgroup. Two canonical exponent are equivalent if and only if $\xi' = \xi + \Delta[\Lambda]$ on some canonical neighborhood, where $\Lambda(r)$ is a linear form in the canonical coordinates of $r$, $\Delta[\Lambda] = \Lambda(r) + \Lambda(s) - \Lambda(rs)$.

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2. To every canonical local exponent of $G$ corresponds uniquely a infinitesimal exponent $\Xi(a,b)$ of the Lie algebra $\mathfrak{g}$ of $G$, i.e., a bilinear antisymmetric form which satisfies the identity $\mathrm{d}\,\Xi(a,a',a'') = 0$. The correspondence is linear.

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3. Two canonical local exponents, $\xi, \xi'$ are equivalent if and only if the corresponding $\Xi, \Xi'$ are equivalent, i.e., if $\Xi'(a,b) = \Xi(a,b) - \Lambda([a,b])$, where $\Lambda(a)$ is a linear form on $\mathfrak{g}$.

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4. There exists a one-to-one linear correspondence between the equivalence classes of local exponents of $G$ and the equivalence classes of infinitesimal exponents of $\mathfrak{g}$.

# Galilei's group in (3 + 1)-dimensions[5]

$G$ group of Galilei's transformations of elements $r, s, t$. The generic transformation of $G$ is given by:

$$\begin{cases} x' = Wx + vt + u \\ t' = t + \eta \end{cases}$$

where $W$ orthogonal transformation, $v, u \in \mathbb{R}^3$, $\eta \in \mathbb{R}$

$$r = (W_r, \eta_r, v_r, u_r)$$
$$rs = (W_r W_s, \eta_r + \eta_s, W_r v_s + v_r, u_r + W_r u_s + \eta_s v_r)$$
$$e = (1, 0, 0, 0), \qquad r^{-1} = \left(W_r^{-1}, -\eta_r, -W_r^{-1} v_r, -W_r^{-1}(u_r - \eta_r v_r)\right)$$

$$Z(r) = \begin{pmatrix} W_r & v_r & u_r \\ 0 & 1 & \eta_r \\ 0 & 0 & 1 \end{pmatrix}$$

$$(W, \eta, v, u) = (1, 0, v, 0) \cdot (W, \eta, 0, u)$$

[5]Bargmann, 1954

# Algebra of $G$

**Infinitesimal generators**:

$a_{ij}$ ⟶orthogonal transformations' generators
$$\begin{pmatrix} M_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$d_j$ ⟶Galilei's pure transformations' generators
$$\begin{pmatrix} 0 & \kappa_j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$b_j$ ⟶space translations' generators
$$\begin{pmatrix} 0 & 0 & \kappa_j \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$f$ ⟶time translations' generators
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

# Algebra of $G$

**Commutations rules**:

$$[a_{ij}, a_{kl}] = \delta_{jk}a_{il} - \delta_{ik}a_{jl} + \delta_{il}a_{jk} - \delta_{jl}a_{ik}$$

$$[a_{ij}, b_k] = \delta_{jk}b_i - \delta_{ik}b_j; \qquad [b_i, b_j] = 0$$

$$[a_{ij}, d_k] = \delta_{jk}d_i - \delta_{ik}d_j; \qquad [d_i, d_j] = 0; \qquad [d_i, b_j] = 0$$

$$[a_{ij}, f] = 0; \qquad [b_k, f] = 0; \qquad [d_k, f] = b_k$$

**Infinitesimal exponents**:
Every infinitesimal exponents are equivalent to **zero** except one:

$$\Xi(b_i, d_k) = -\Xi(d_k, b_i) = \gamma\delta_{ik}, \qquad \gamma \in \mathbb{R}$$

**Calculation of group's exponents** ($\hbar = 1$):
The equivalence classes of non equivalent exponents are multiples of a given $\xi_0(r, s)$, with $\gamma$ multiplicative factor:

$$\xi_0(r, s) = \frac{1}{2}\left(\langle u_r \mid W_r v_s\rangle - \langle v_r \mid W_r u_s\rangle + \langle \eta_s v_r \mid W_r v_s\rangle\right)$$

$\xi_0$ bilinear form which holds definition relations

# Generators of $G$

**Generators of Galilei's group**[6]:

$$a_{ij} \rightarrow M_k = i\epsilon^{ijk} a_{ij} \qquad [M_i, M_j] = i\epsilon^{ijk} M_k$$
$$d_j \rightarrow D_j = i d_j \qquad [M_i, D_j] = i\epsilon^{ijk} D_k;\ [D_i, D_j] = 0$$
$$b_j \rightarrow B_j = i b_j \qquad [M_i, B_j] = i\epsilon^{ijk} B_k;\ [B_i, B_j] = 0$$
$$f \rightarrow H = i f \qquad [M_i, H] = 0;\ [B_k, H] = 0;\ [D_k, H] = i B_k$$

$$\Xi(b_i, d_j) = 2\xi(b_i, d_j) = \gamma\delta_{ij}, \qquad \xi(b_i, d_j) = \frac{1}{2}\sum \beta_{kl}\rho_k^{(i)}\sigma_l^{(j)}$$

$$\mathrm{e}^{i\beta D_j}\,\mathrm{e}^{i\alpha B_i} = \mathrm{e}^{-i\gamma\alpha\beta\delta_{ij}}\,\mathrm{e}^{i\alpha B_i}\,\mathrm{e}^{i\beta D_j} \quad \Rightarrow \quad [D_i, B_j] = -i\gamma\delta_{ij}$$

$\exists S$ such that $SD_iS^{-1} = \hat{D}_i$, $SQ_iS^{-1} = Q_i$, $SP_iS^{-1} = P$ e $\hat{D}_i - \gamma Q_i = 0$. Then:

$$H = H_0 + V(Q), \qquad H_0 = \frac{P^2}{2\gamma}$$

with $\gamma \equiv$ particle mass: for every particle of different mass exists a non-equivalent multiplicator and so a different ray representation of Galilei's group[7]

[6] Bose, 1995; Toller, 1999

[7] Brennich, 1970

# Galilei's group in quantum mechanics[8]

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2\mu}\nabla^2\psi = 0$$

Let be $\psi$ solutions of Schrödinger equation, $r$ element of $G$, Galilei's group:

$$\psi(X) \rightarrow \psi'(X') = \mathrm{e}^{-i\theta(r,X')}\,\psi(r^{-1}X)$$

$$\theta(r,X) = \mu\left(\frac{1}{2}t\,\langle v\,|\,v\rangle - \langle v\,|\,x\rangle\right)$$

**Phase exponent**

$$\xi_q(r,s) = \langle u_r\,|\,W_r v_s\rangle - \frac{\eta_r}{2}\langle v_s\,|\,v_s\rangle - \eta_r\,\langle v_r\,|\,W_r v_s\rangle$$

$$\xi_q = \xi_0 + \Delta[\zeta], \qquad \zeta(r) = \frac{1}{2}\left(\eta_r\,\langle v_r\,|\,v_r\rangle - \langle v_r\,|\,u_r\rangle\right)$$

**Representation of $G$**

$$\phi'(k) = U_r\phi(r^{-1}k) = \mathrm{e}^{-i(\langle k\,|\,u\rangle - \frac{\eta}{2\gamma}\langle k\,|\,k\rangle + \frac{\eta}{2}\gamma\langle v\,|\,v\rangle - \gamma\langle u\,|\,v\rangle)}\,\phi(r^{-1}k)$$

[8] Bargmann, 1954

# Galilei's group in $(2+1)$-dimensions[10]

### Phase exponents

$$\xi_1(r,s) = \frac{1}{2}D(v_r, W_r v_s) \equiv \frac{1}{2}(v_r \wedge W_r v_s), \qquad \Xi(d_i, d_j) \not\equiv 0,\ \lambda$$

$$\xi_2(r,s) = \theta_r \eta_s - \theta_s \eta_r \qquad \Xi(a_{ij}, f) = S$$

### Semi-direct product of $G$: $G^{\gamma\lambda S} = A^{\gamma S} \times_t H^\lambda$

$$\begin{aligned} A^{\gamma S} &= \{(1,\eta,0,u;\zeta_0,1,\zeta_2)\} \\ H^\lambda &= \{(W,0,v,0;1,\zeta_1,1)\} \end{aligned} \rightarrow (W,\eta,v,u;\zeta_0,\zeta_1,\zeta_2) = (1,\eta,0,u;\zeta_0',1,\zeta_2')\cdot(W,0,v,0;1,\zeta_1,1)$$

$$\zeta_0' = \zeta_0\, \mathrm{e}^{-i\frac{\gamma}{2}\langle u\,|\,v\rangle}, \qquad \zeta_2' = \zeta_2\, \mathrm{e}^{iS\theta\eta}$$

Representation of $G$: $U(g) = \chi^9(a)V(h)$, $a \in A^{\gamma S}$, $h \in H^\lambda$

**Ray representations:** $\gamma = \lambda = S = 0$

$$U(W,\eta,v,u) = \mathrm{e}^{ip_0\eta}\,\pi(W,0,v,0)$$

$$(U(W,\eta,v,u)f)(p_0,p) = \mathrm{e}^{i(\eta p_0 + \langle u\,|\,p\rangle)}\,\mathrm{e}^{ia\frac{p\wedge v}{r}}\, f(p_0 + \langle v\,|\,p\rangle, W^{-1}p)$$

[9] Karpilovsky, 1994
[10] Grigore, 1996; Bose, 1995

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**Ray representations:** $\gamma \neq 0$, $\lambda \neq 0$, $S = 0$

$$U(W,\eta,v,u)f)(p) = \mathrm{e}^{i\left(\langle u\,|\,p\rangle + \frac{\gamma}{2}\langle u\,|\,v\rangle + \frac{\eta}{2\gamma}\langle p\,|\,p\rangle - \frac{\lambda}{2\gamma}(v\wedge p) + s\theta\right)} f(W^{-1}(p+\gamma v))$$

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# Galilei's group in $(1+1)$-dimensions[11]

$$r = (u_r, v_r, \eta_r)$$

**Phase exponent:**

$$\xi_\eta(r,s) = \frac{a_1}{2}(a_r v_s - a_s v_r + \eta_r v_r v_s) + \frac{a_2}{2}(u_r \eta_s - u_s \eta_r - \eta_r \eta_s v_r)$$
$$U_r U_s = \mathrm{e}^{i\xi_\eta(r,s)}\, U_{rs}$$

**Time depending generators**

$$R_t(H) = -\frac{\hbar^2}{2m}\partial_x^2 + fx + V_0$$
$$R_t(P) = -i\hbar\partial_x - ft$$
$$R_t(N) = mx - i\hbar t\partial_x - \frac{1}{2}ft^2$$

$\rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}R_t(X) = KR_t([H,X]), R_{t=0}(X) = R(X)$$
$$V_0 = \frac{a_3}{2a_1} = R(H) - a_2 x + \frac{\hbar^2}{2a_1}\partial_x^2$$
$$C_3 = 2HZ_1 - 2KZ_2 - P^2, m = a_1, f = a_2$$

[11] Doebner, Mann, 1995

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# Extension of Doebner and Mann approach

Doebner-Mann → higher dimensions

$$R_t(H) = R(H),\ R_t(P_i) = R(P_i),\ R_t(M) = R(M)$$
$$R_t(N_i) = -iR(P_i)t + R(N_i)$$

**$\gamma \neq 0, \lambda \neq 0, S = 0$**

$$N_1 = -ip_1 t + \gamma \frac{\partial}{\partial p_1} - i\frac{\lambda}{2\gamma} p_2$$
$$N_2 = -ip_2 t + \gamma \frac{\partial}{\partial p_2} - i\frac{\lambda}{2\gamma} p_1$$

**$\gamma \neq 0, \lambda = S = 0$**

$$N_i = -ip_i t + \mu \frac{\partial}{\partial p_i}$$

$$(U_t(r)f)(p) = \mathrm{e}^{-i\langle p \mid v_r \rangle t}(U(r)f)(p)$$
$$(U_t(r)U_t(s)f)(p) = \phi(r,s,t)\omega(r,s)(U_t(rs)f)(p)$$
$$\phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \mid W_r v_s \rangle t}$$

# Physical interpretation of the phase exponents

## Bargmann's phase exponent

$$\xi_0(r,s) = \frac{1}{2}\left(\langle u_r \,|\, W_r v_s\rangle - \langle v_r \,|\, W_r u_s\rangle + \langle \eta_s v_r \,|\, W_r v_s\rangle\right)$$

$$\begin{cases} r = (1,0,0,u) \\ s = (1,0,v,0) \end{cases} \to U(r)U(s) = \mathrm{e}^{i\xi(r,s)}\, U(rs) \to \begin{cases} \xi(r,s) = \frac{\gamma}{2}\langle u \,|\, v\rangle \\ rs = (1,0,v,u) \end{cases}$$

# Physical interpretation of the phase exponents

## Time depending phase exponent

$$\phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \mid W_r v_s\rangle\, t}$$

$$\begin{cases} r = (1,0,v_r,0) \\ s = (1,0,v_s,0) \end{cases} \rightarrow (U_t(r)U_t(s)f)(p) = \mathrm{e}^{i\xi(r,s)}\,\phi(r,s,t)U(rs) \rightarrow$$

$$\rightarrow \begin{cases} \phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \mid v_s\rangle t} \\ rs = (1,0,v_r+v_s,0) \end{cases}$$

# Physical interpretation of the phase exponents

## Time depending phase exponent

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$$\Sigma \underset{v_0}{\longrightarrow} \Sigma'$$

$$\frac{1}{2}\gamma v'^2 = \frac{1}{2}\gamma v^2 + \frac{1}{2}\gamma v_0^2 + \gamma\,\langle\, v \mid v_0\,\rangle$$

# Physical interpretation of the phase exponents

## Time depending phase exponent

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$$\to \begin{cases} \phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \mid v_s\rangle t} \\ rs = (1,0,v_r+v_s,0) \end{cases}$$

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# Physical interpretation of the phase exponents

## Time depending phase exponent

$$\phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \,|\, W_r v_s\rangle\, t}$$

$$\begin{cases} r =(1,0,v_r,0) \\ s =(1,0,v_s,0) \end{cases} \longrightarrow (U_t(r)U_t(s)f)(p) = \mathrm{e}^{i\xi(r,s)}\,\phi(r,s,t)U(rs) \longrightarrow$$

$$\longrightarrow \begin{cases} \phi(r,s,t) = \mathrm{e}^{-i\gamma\langle v_r \,|\, v_s\rangle t} \\ rs =(1,0,v_r+v_s,0) \end{cases}$$

$$\Sigma \xrightarrow{r} \Sigma_r \xrightarrow{s} \Sigma_s$$

$$\gamma\langle v_r \,|\, v_s\rangle = (\gamma\langle v \,|\, v_0\rangle)_{rs} - \left[(\gamma\langle v \,|\, v_0\rangle)_r + (\gamma\langle v \,|\, v_0\rangle)_s\right]$$

# Conclusion

Application of Bargmann's ray representation theory to Galilei's group in:

- $(3+1)$-dimensions;
- $(2+1)$-dimensions;
- $(1+1)$-dimensions.
    - Extension of Doebner and Mann approach to higher dimensions;
    - Calculation of time depending ray representation of Galilei's group in $(2+1)$- and $(3+1)$-dimensions.

Future developments:

- In order to develop phisycal application of Wawrzycki's generalization of Bargmann's theory[12], we'll study the possibility to extend the Doebner and Mann approach to Poincaré group.

[12] Wawrzycki, 2004